Homework 1 Sample Solutions

provided by Alex Grounds

Extra Exercise 1. Show that for $a, b \in (0, \infty)$,

$$\ln a - \ln b = \ln \frac{a}{b}$$

Solution. Let's define $z = \ln a - \ln b$. Exponentiating, we get $e^z = e^{\ln a - \ln b}$. Using the facts that $e^{a+b} = e^a e^b$ and $e^{-a} = \frac{1}{e^a}$, we obtain

$$e^z = e^{\ln a - \ln b} = e^{\ln a} e^{-\ln b} = \frac{e^{\ln a}}{e^{\ln b}}.$$

Of course, since e^x and $\ln x$ are inverse functions, it follows that

$$e^z = \frac{e^{\ln a}}{e^{\ln b}} = \frac{a}{b}.$$

Now all we have to do is take the natural log of both sides:

$$\ln e^z = z = \ln \frac{a}{b}.$$

But since we defined $z = \ln a - \ln b$, we have exactly what we wanted!

$$\ln a - \ln b = z = \ln \frac{a}{b}$$

as claimed. Note that we have implicitly used that e^x is a one-to-one function here. After all, if it were not, it would not have a well-defined inverse function (but it does, namely $\ln x$). $\square$

Extra Exercise 2. Show that

$$\frac{d}{dx} \tan^{-1} x = \frac{1}{1 + x^2}$$

Here are the steps. Do each one of them: (a), (b), (c) are unrelated to each other, but they all get combined in part (d) to prove the result.

(a) Suppose that f and g are two differentiable functions that are inverses of each other. Show that

$$g'(x) = \frac{1}{f'(g(x))}$$

(if $f'(g(x)) \neq 0$, of course. Question: could $f'(g(x))$ be 0? Think of an example or why not)

(b) Show that

$$\frac{d}{dx}\tan x = \sec^2 x,$$

where recall that, by definition $\sec x = \frac{1}{\cos x}$.

(c) Show that

$$\sec^2(\tan^{-1} x) = 1 + x^2.$$

(d) Combine (a), (b) and (c) to draw the conclusion.

Solution.

(a) Consider the function $f(g(x))$. Since f and g are inverse functions, we have $f(g(x)) = x$ (this is the definition of inverse functions). If we differentiate each side of this equation, we get the following:

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x) = \frac{d}{dx}x = 1$$

The first equality is simply the chain rule. Since we have assumed that $f'(g(x)) \neq 0$, we can divide by it to obtain

$$g'(x) = \frac{1}{f'(g(x))}$$

as claimed.

Note that it looks like this only works in the special case that $f'(g(x)) \neq 0$. But this is not actually a special case at all. After all, we have that

$$f'(g(x))g'(x) = 1$$

(as we showed above). Thus, if $f'(g(x))$ were 0, then we would have

$$f'(g(x))g'(x) = 0g'(x) = 0 = 1?!$$

which is obviously a contradiction.

NOTE BY MONA:
Note that if I had not specified that f and g are both differentiable the above argument would break.

Think of the following example: $f(x) = x^3$: this is a continuous differentiable function with continuous derivative $f'(x) = 3x^2$ and continuous inverse $g(x) = \sqrt[3]{x}$. Then $f'(g(x)) = 3(\sqrt[3]{x})^2$, which is clearly 0 for $x = 0$.

Obviously, in this example we could not have $f'(g(x))g'(x) = 1$. Where the argument above that shows that $f'(g(x))g'(x) = 1$ breaks is the application of the chain rule: the chain rule only holds for the points x for which both g and f are differentiable. In this example, g is *not* differentiable at $x = 0$.

However, in the question above you were able to apply the chain rule to $f(g(x))$ and conclude that $f'(g(x))g'(x) = 1$, but that's only because we assumed that both f and g are differentiable for any x.

(b) We wish to compute $\frac{d}{dx}\tan x$. Using the quotient rule, we have that

$$\frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x} = \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

Here we used the trigonometric identity $\sin^2 x + \cos^2 x = 1$ for any x.

(c) Let's set $y = \tan^{-1} x$. By definition, this means that $\tan y = x$. We want to show that $\sec^2 y = 1 + x^2$.

Note that

$$\begin{aligned} \tan^2 y + 1 &= \frac{\sin^2 y}{\cos^2 y} + \frac{\cos^2 y}{\cos^2 y} && \text{we used } \tan y = \frac{\sin y}{\cos y} \text{ and } 1 = \frac{\cos^2 y}{\cos^2 y} \\ &= \frac{\sin^2 y + \cos^2 y}{\cos^2 y} \\ &= \frac{1}{\cos^2 y} && \text{since } \sin^2 y + \cos^2 y = 1 \text{ for any } y \\ &= \sec^2 y \end{aligned}$$

So we have shown that

$$\tan^2 y + 1 = \sec^2 y$$

for all y. Now, if we simply plug $y = \tan^{-1} x$ into this equation, we get

$$\tan^2 y + 1 = \tan^2(\tan^{-1} x) + 1 = (\tan(\tan^{-1}(x)))^2 + 1 = x^2 + 1$$

$$= \sec^2 y = \sec^2(\tan^{-1} x)$$

as claimed. Here, we used that $\tan x$ and $\tan^{-1} x$ are inverse functions (by definition).

(d) Putting all of this together, we get that

$$\frac{d}{dx}\tan^{-1} x \overset{\text{by (a)}}{=} \frac{1}{(\tan)'(\tan^{-1}(x))} \overset{\text{by (b)}}{=} \frac{1}{\sec^2(\tan^{-1}(x))} \overset{\text{by (c)}}{=} \frac{1}{1+x^2}$$

as claimed.

□

Exercise 7.1 #32. Use substitution to evaluate

$$\int \sin^3 x \cos x \, dx.$$

Solution. We make the substitution $u = \sin x$, $du = \cos x \, dx$, so we get

$$\int \sin^3 x \cos x \, dx = \int u^3 \, du = \frac{1}{4}u^4 + C = \frac{1}{4}\sin^4 x + C.$$

□

Exercise 7.1 #42. Assuming $g(x)$ is a continuous function whose derivative $g'(x)$ is also continuous, use substitution to evaluate

$$\int \frac{g'(x)}{[g(x)]^2 + 1} \, dx.$$

Solution. We make the substitution $u = g(x)$, $du = g'(x) \, dx$ (this works because $g(x)$ is continuous with a continuous derivative). Thus, we get

$$\int \frac{g'(x)}{[g(x)]^2 + 1} \, dx = \int \frac{du}{u^2 + 1} = \tan^{-1}(u) + C = \tan^{-1}(g(x)) + C.$$

Here we used Extra Exercise #2 to recognize $\tan^{-1}(u)$ as an antiderivative of $\frac{1}{u^2+1}$. □